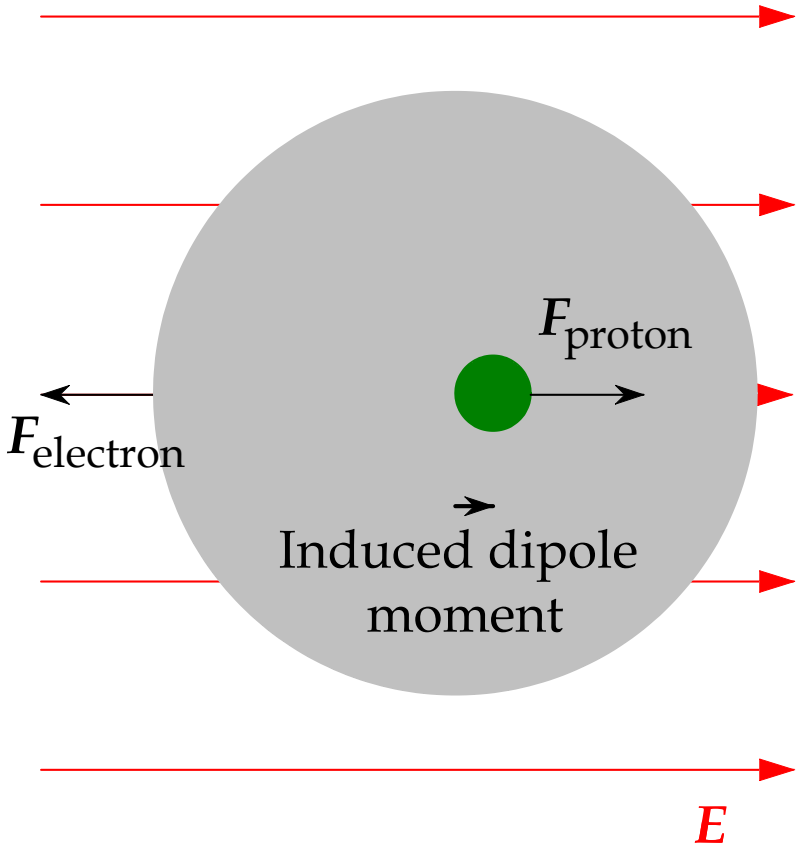
Today in Physics 217: electric dipoles and their interactions

- Origin of coordinates for multipole moments
- Force, torque, and potential energy for dipoles in uniform electric fields
- Force on dipoles in nonuniform electric fields
- Example: dipole *vs*. dipole
- Permanent and induced dipoles in atoms and molecules



Choice of coordinate origin matters a lot in multipole expansion

Changing the origin doesn't change the physics, but it can radically change the terms in the series. Consider a point charge at the origin:

$$V = \frac{q}{r} \quad \text{Monopole only, of course}$$

and one not at the origin, say at $(r,\theta,\phi) = (a,\theta_0,0)$:

$$V = \frac{q}{r}\left[1 + \frac{a}{r}\cos\theta_0 + \left(\frac{a}{r}\right)^2 P_2(\cos\theta_0) + \left(\frac{a}{r}\right)^3 P_3(\cos\theta_0) + \ldots\right]$$

How did a point charge get all these nonzero multipole moments? It didn't, really; the situation still amounts to one point charge, but the accounting is more complicated, when $1/\boldsymbol{\imath} \neq 1/r$.

Choice of coordinate origin matters a lot in multipole expansion (continued)

- So when someone gives you a charge distribution and asks what all the moments are, they also have to tell you what coordinate system to use.
- One useful exception: if the total charge is zero, then the monopole moment is zero, and the dipole moment is independent of the choice of coordinate origin.

  Consider such a situation, and suppose the dipole moment were $\boldsymbol{p}$ originally and $\boldsymbol{p}_a$ in a coordinate system with its origin displaced by some vector $\boldsymbol{a}$:

$$\boldsymbol{p}_a = \int \boldsymbol{r}'_a \rho(\boldsymbol{r}')\, d\tau' = \int (\boldsymbol{r}' - \boldsymbol{a})\, \rho(\boldsymbol{r}')\, d\tau'$$

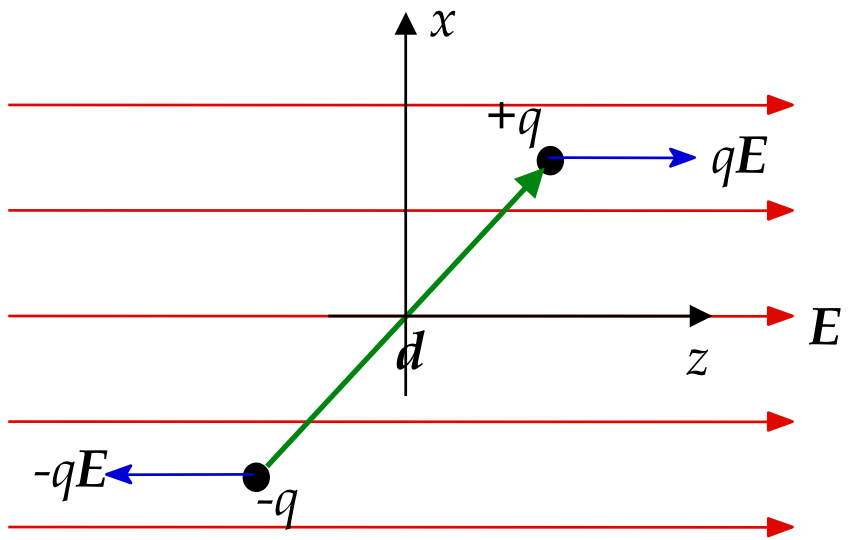
$$= \int \boldsymbol{r}' \rho(\boldsymbol{r}')\, d\tau' - \boldsymbol{a}\int \rho(\boldsymbol{r}')\, d\tau' = \boldsymbol{p} - \boldsymbol{a}\,\underbrace{q}_{0} = \boldsymbol{p} \quad .$$

Force and torque on dipole in uniform E field

If the dipole moment is constant, the net force is zero, because the charges get pulled equally and oppositely.

There is a *torque*, though, that tends to align the dipole moment vector with the applied field:

$$\begin{aligned}\boldsymbol{N} &= \boldsymbol{r}_+ \times \boldsymbol{F}_+ + \boldsymbol{r}_- \times \boldsymbol{F}_- \\ &= \frac{\boldsymbol{d}}{2} \times q\boldsymbol{E} + \left(-\frac{\boldsymbol{d}}{2}\right) \times \left(-q\boldsymbol{E}\right) \\ &= q\boldsymbol{d} \times \boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E} \\ &= -pE\hat{\boldsymbol{y}} \quad \text{, in this case.}\end{aligned}$$



Potential energy of dipole in uniform *E* field (Griffiths problem 4.7)

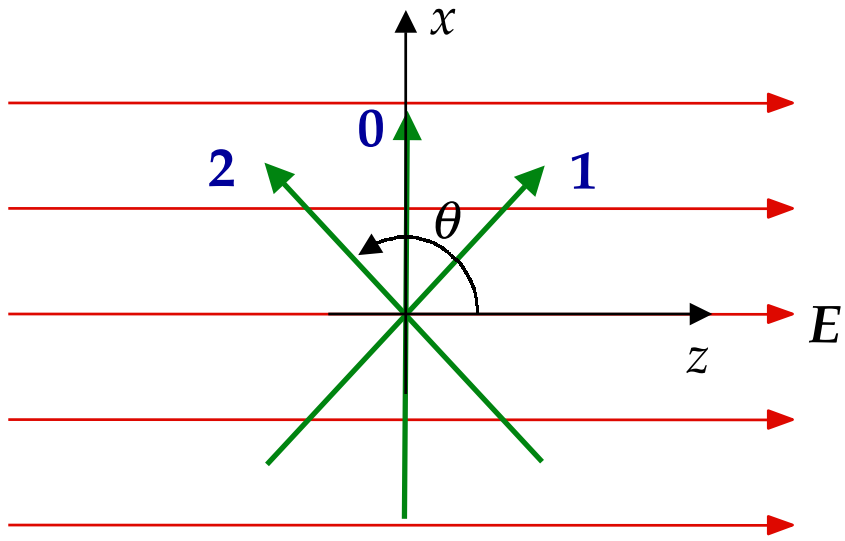
Consider a dipole initially perpendicular to the field (**0**). The field tends to pull it into alignment (toward **1**); we have to push to make it move toward **2**. Thus the work we do is

$$W = \int_{\pi/2}^{\theta} \left| \boldsymbol{p} \times \boldsymbol{E} \right| d\theta'$$

$$= pE \int_{\pi/2}^{\theta} \sin\theta' d\theta'$$

$$= -pE\cos\theta = -\boldsymbol{p}\cdot\boldsymbol{E}$$

$$U = \quad .$$



(Work *we* do = potential energy.)

Force on dipole in nonuniform *E* field

If *E* changes with position, the forces on each charge in a simple dipole will no longer in general be equal in magnitude, leading to a net force ***F*** on the dipole. Suppose the dipole is very short; infinitesimal, in fact:

$$\boldsymbol{F} = q\left(\boldsymbol{E}_+ - \boldsymbol{E}_-\right) = q\Delta\boldsymbol{E}$$

Definition of gradient:

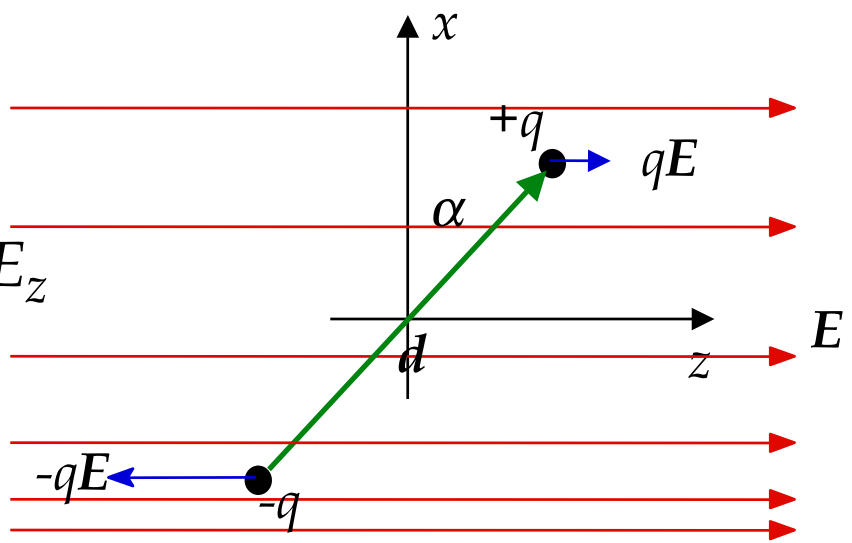
$$\frac{\Delta E_z}{d\cos\alpha} = \left(\nabla E_z\right)_x \quad \Rightarrow \Delta E_z = \boldsymbol{d}\cdot\nabla E_z$$

$$\Delta E_x = \boldsymbol{d}\cdot\nabla E_x$$

$$\Delta E_y = \boldsymbol{d}\cdot\nabla E_y$$

(if we had chosen to give ***E*** a *y* or an *x* component)

$$\therefore \boldsymbol{F} = q\left(\boldsymbol{d}\cdot\nabla\right)\boldsymbol{E} = \left(\boldsymbol{p}\cdot\nabla\right)\boldsymbol{E}$$



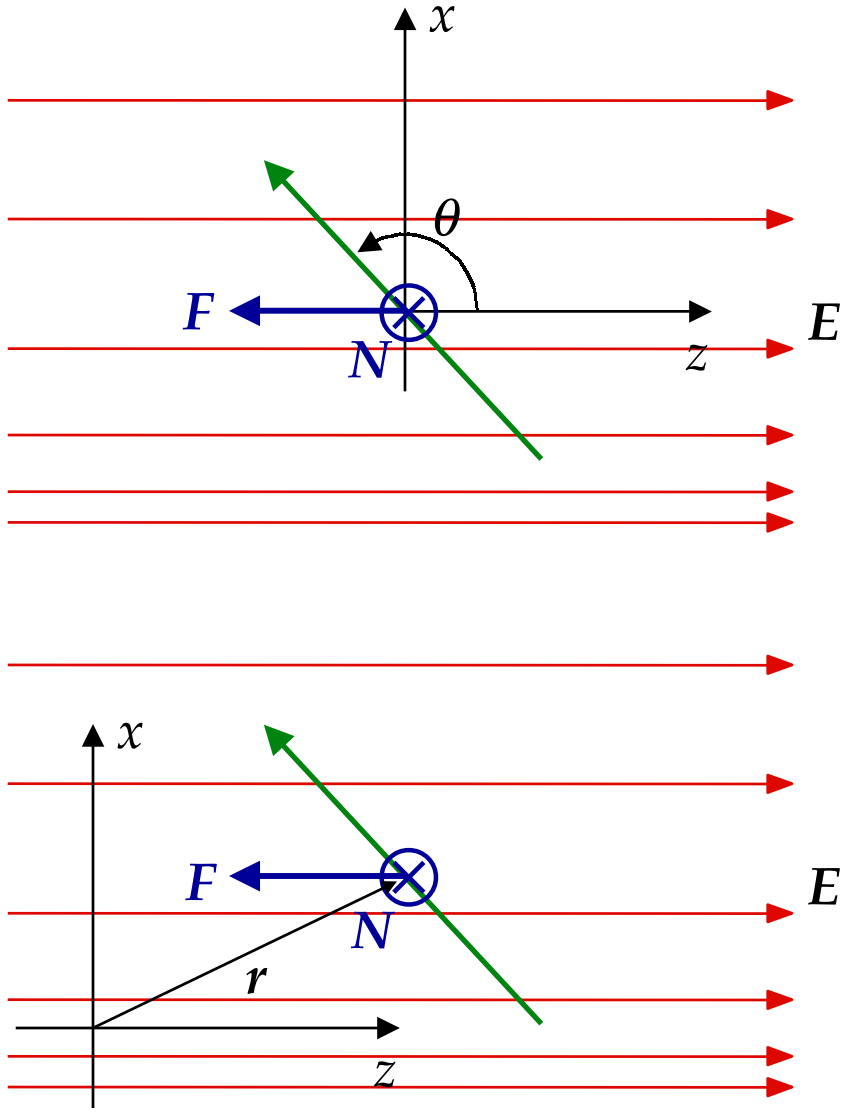
Torque on dipole in nonuniform *E* field

The torque on a *small* dipole, about its own center, is still

$$\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E}$$

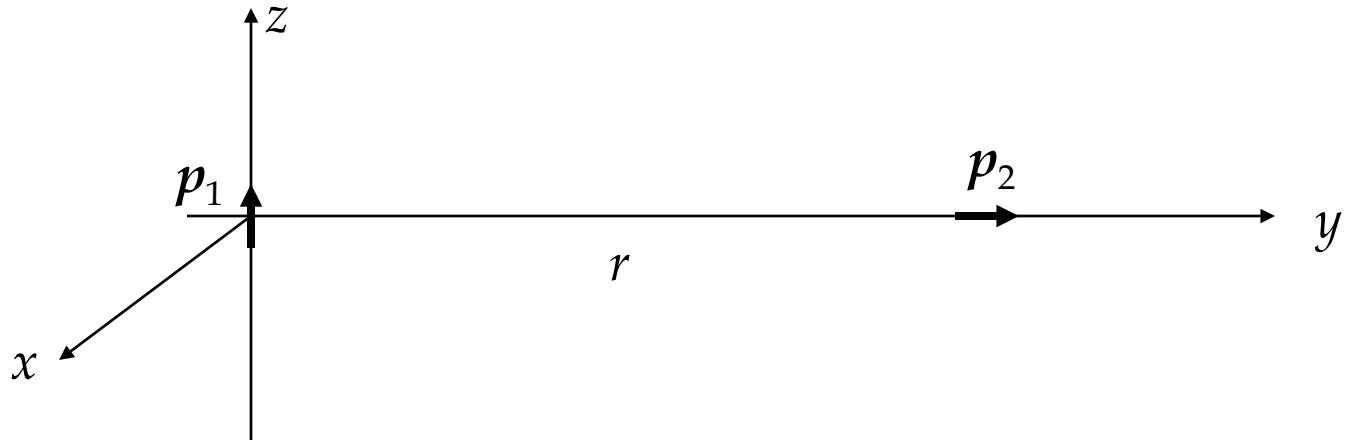
since the same arguments apply as before. But since there's a net force $\boldsymbol{F}$ on the dipole, there's an extra torque of $\boldsymbol{r} \times \boldsymbol{F}$ about any other point:

$$\boldsymbol{N} = \boldsymbol{p} \times \boldsymbol{E} + \boldsymbol{r} \times \boldsymbol{F}$$



Dipole *vs.* dipole: force and torque

Griffiths problems 4.5 and 4.29: Two perfect (infinitesimal) dipoles $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are perpendicular and lie a distance r apart. What is the torque on $\boldsymbol{p}_1$ (about its center) due to $\boldsymbol{p}_2$? What is the torque on $\boldsymbol{p}_2$ (about its center) due to $\boldsymbol{p}_1$? What are the forces on each, due to each? Why are the torques not equal and opposite?



Dipole *vs.* dipole: force and torque (continued)

The field of each, at each other's position:

$$\boldsymbol{E}_1(2) = \frac{2p_1\cos\theta}{r^3}\hat{\boldsymbol{r}} + \frac{p_1\sin\theta}{r^3}\hat{\boldsymbol{\theta}} = \frac{p_1}{r^3}\hat{\boldsymbol{\theta}} = -\frac{p_1}{y^3}\hat{\boldsymbol{z}} = -\frac{p_1}{a^3}\hat{\boldsymbol{z}}$$

$$\boldsymbol{E}_2(1) = \frac{2p_2\cos\theta'}{r'^3}\hat{\boldsymbol{r}}' + \frac{p_2\sin\theta'}{r'^3}\hat{\boldsymbol{\theta}}' = -\frac{2p_2}{r'^3}\hat{\boldsymbol{r}}' = \frac{2p_2}{y^3}\hat{\boldsymbol{y}} = \frac{2p_2}{a^3}\hat{\boldsymbol{y}}$$

The torque on each:

$$\boldsymbol{N}_1 = \boldsymbol{p}_1 \times \boldsymbol{E}_2(1) = p_1\hat{\boldsymbol{z}} \times \frac{2p_2}{a^3}\hat{\boldsymbol{y}} = \boxed{-\frac{2p_1p_2}{a^3}\hat{\boldsymbol{x}}}$$

$$N_2 = \boldsymbol{p}_2 \times \boldsymbol{E}_1(2) = p_2\hat{\boldsymbol{y}} \times \left(-\frac{p_1}{a^3}\hat{\boldsymbol{z}}\right) = \boxed{-\frac{p_1p_2}{a^3}\hat{\boldsymbol{x}}}$$

Dipole *vs.* dipole: force and torque (continued)

Force on 2 due to field of 1:

$$\boldsymbol{F}_2 = (\boldsymbol{p}_2 \cdot \nabla)\boldsymbol{E}_1(2) = p_2 \frac{\partial}{\partial y}\boldsymbol{E}_1(2)\Big|_{y=a}$$

$$= p_2 \frac{\partial}{\partial y}\left(-\frac{p_1}{y^3}\hat{\boldsymbol{z}}\right)\Big|_{y=a} = \boxed{\frac{3p_1p_2}{a^4}\hat{\boldsymbol{z}}} \quad .$$

By Newton's third law, the force on 2 by 1 is

$$\boldsymbol{F}_1 = (\boldsymbol{p}_1 \cdot \nabla)\boldsymbol{E}_2(1) = \boxed{-\frac{3p_1p_2}{a^4}\hat{\boldsymbol{z}}} \quad .$$

Dipole *vs.* dipole: force and torque (continued)

The forces the dipoles exert on each other are equal and opposite. Why aren't the torques? Because we calculated the torques about different centers. If we refer both torques to the coordinate origin (i.e. the position of dipole 1), then

$$\boldsymbol{N}_1(0) = \boldsymbol{p}_1 \times \boldsymbol{E}_2(1) = -\frac{2p_1p_2}{a^3}\hat{\boldsymbol{x}} \quad \text{(still)}$$

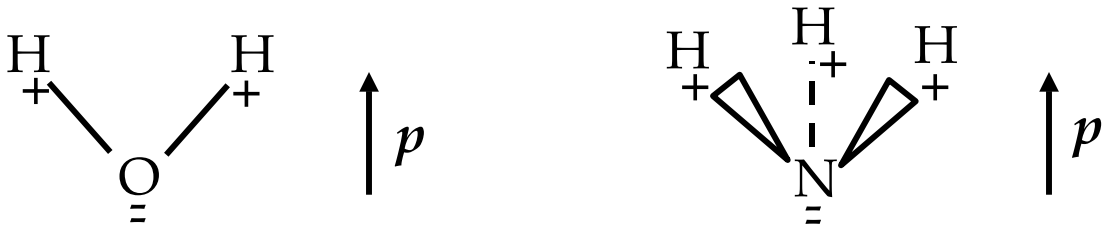
$$N_2(0) = \boldsymbol{p}_2 \times \boldsymbol{E}_1(2) + \boldsymbol{r} \times \boldsymbol{F}_2 = -\frac{p_1p_2}{a^3}\hat{\boldsymbol{x}} + a\hat{\boldsymbol{y}} \times \frac{3p_1p_2}{a^4}\hat{\boldsymbol{z}}$$

$$= -\frac{p_1p_2}{a^3}\hat{\boldsymbol{x}} + \frac{3p_1p_2}{a^3}\hat{\boldsymbol{x}} = \frac{2p_1p_2}{a^3}\hat{\boldsymbol{x}} = \boxed{-\boldsymbol{N}_1(0)} \quad ,$$

as you always thought it should be.

Permanent and induced dipoles in atoms and molecules

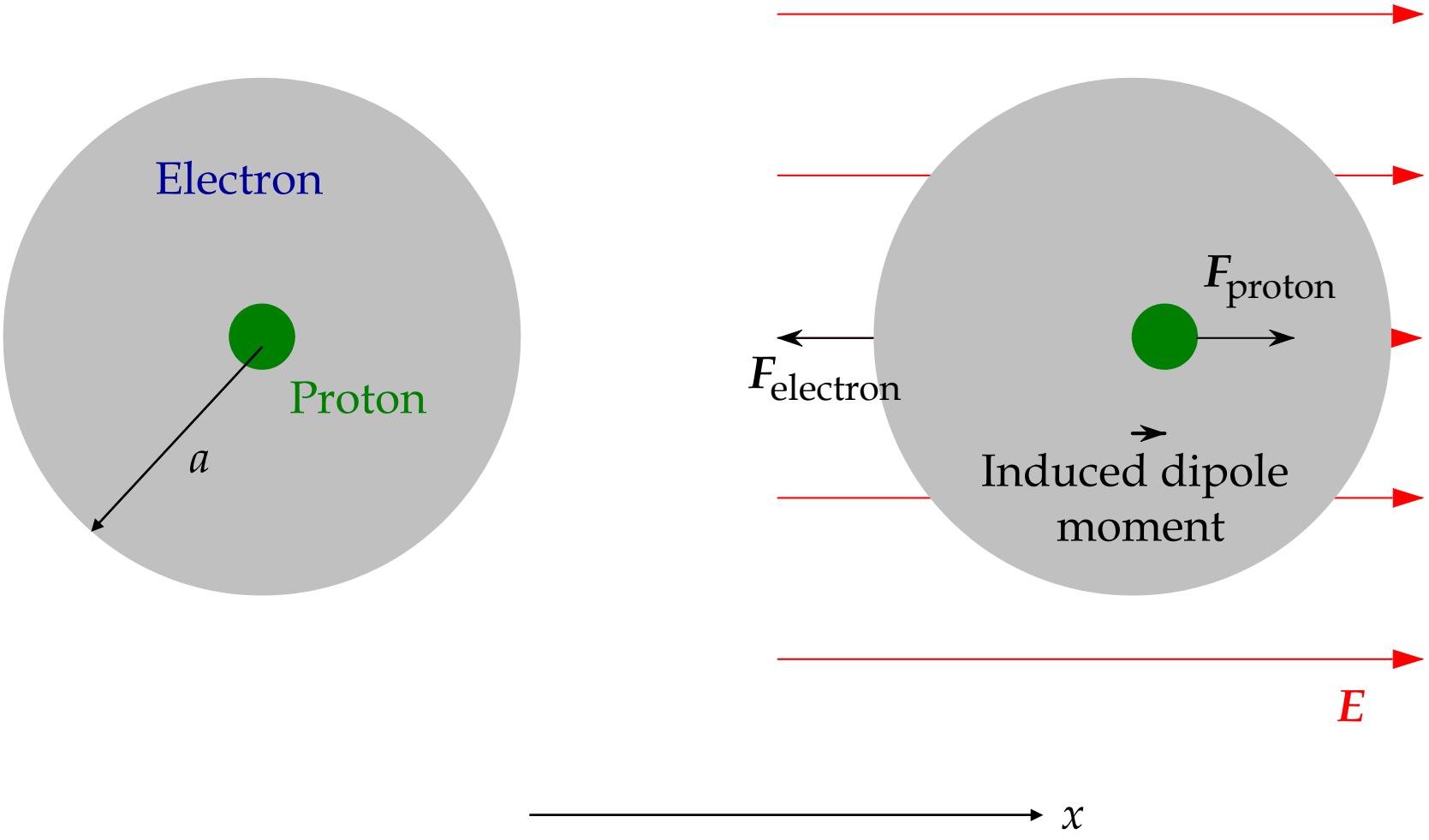
Some naturally-occurring charge distributions, such as many simple molecules, have permanent, built-in dipole moments:



and the electrical properties of *materials* made with these components are decisively influenced by the behaviour of these dipoles in each others' fields.

Some materials are of course made up of neutral, non-polar components, but even these can have dipole moments *induced* by external fields. Consider even the lowly hydrogen atom:

Permanent and induced dipoles in atoms and molecules (continued)



Permanent and induced dipoles in atoms and molecules (continued)

The electron and proton move in opposite directions until the force on the proton by the displaced electron balances the force on the proton by the external field. Suppose (crudely) that the electron has uniform charge density and stays spherical through this process. If the equilibrium position of the proton is a distance d from the center of the electron, then

$$\boldsymbol{E}_{-q}(q) = -\hat{\boldsymbol{x}}\frac{1}{d^2}q\frac{\frac{4}{3}\pi d^3}{\frac{4}{3}\pi a^3} = -\hat{\boldsymbol{x}}\frac{qd}{a^3} = -\hat{\boldsymbol{x}}\frac{p_{\text{induced}}}{a^3} = -\boldsymbol{E}_{\text{external}},\ \text{or}$$

$$\boldsymbol{p}_{\text{induced}} = \alpha \boldsymbol{E}_{\text{external}},\quad \alpha = a^3 \quad \left(= 4\pi\varepsilon_0 a^3 \text{ in MKS}\right).$$

α, called the **polarizability**, is thus proportional to atomic volume.

Atomic polarizabilities

Element	$\alpha\left(10^{-24}\ \text{cm}^{-3}\right)$
H	0.66
Li	12.0
Na	27
K	34